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Generalized Stability of Isometries

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Let X and Y be real Banach spaces and let $\varepsilon, p \geq 0$. A mapping f: $X \to Y$ is called an (ε, p)-isometry if $|\|f(x) - f(y)\| - \|x - y\|| \leq \varepsilon \|x - y\|^p$ holds for all $x, y \in X$. A pair (X, Y) is p-stable with respect to isometries if there exists a function δ: $[0, \infty) \to [0, \infty)$ with $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$ such that for every surjective (ε, p)-isometry f: $X \to Y$ there is a surjective isometry U: $X \to Y$ satisfying the estimate $\|f(x) - U(x)\| \leq \delta(\varepsilon)\|x\|^p$, $x \in X$. We show that every pair of Banach spaces (X, Y) is p-stable for $0 \leq p < 1$. The pair $(\mathbb{R}^2, \mathbb{R}^2)$ is not 1-stable. When $p > 1$ a superstability phenomenon occurs for finite-dimensional Banach spaces.

Key Words: (ε, p)-isometry; stability of isometries.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let X and Y be real Banach spaces and let $\varepsilon \geq 0$. A mapping f: $X \to Y$ is called an ε-isometry if $|\|f(x) - f(y)\| - \|x - y\|| \leq \varepsilon$. In 1945 Hyers and Ulam [6] posed a stability problem for isometries, asking whether for each surjective ε-isometry f there exists an isometry U: $X \to Y$ that is close to f. They gave a positive answer in the case when $X = Y$ is a real Hilbert space and observed that the surjectivity assumption on f is essential. The problem was solved by Gevirtz [4], who essentially used a result of Gruber [5]. He showed that every surjective ε-isometry f satisfying $f(0) = 0$ can be uniformly approximated by a linear isometry U such that $\|f(x) - U(x)\| \leq 5\varepsilon$ for every $x \in X$. Omladič and Šemrl [11] showed that the estimate 5ε can be improved to 2ε and that this estimate is sharp.

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Note that when studying ε-isometries there is no loss of generality in assuming that $f(0) = 0$. Indeed, if a mapping f is an ε-isometry then so is $f - f(0)$, and $f - f(0)$ can be approximated by an isometry U if and only if f can be approximated by the isometry $U + f(0)$. So, it will be assumed throughout this paper that each ε-isometry sends the origin into the origin. Later on we will give a more general definition of approximate isometries but the same argument still remains true.

Lindenstrauss and Szankowski [9] studied a wider concept of approximate isometries. For a given surjective mapping f from a real Banach space X onto a real Banach space Y they considered the function

$$\varphi_f(t) = \sup\{|\,\|f(x) - f(y)\| - \|x - y\|\,| : \|x - y\| \le t \text{ or } \|f(x) - f(y)\| \le t\} \tag{1}$$

for $t \ge 0$. They proved that if

$$\int_1^\infty \frac{\varphi_f(t)}{t^2}\, dt < \infty$$

then there is an isometry U from X onto Y, defined by $U(x) = \lim_{n\to\infty}(f(2^n x)/2^n)$, such that

$$\|f(x) - U(x)\| = o(\|x\|) \qquad \text{as } \|x\| \to \infty.$$

Roughly speaking, we can say that they studied a concept of approximate isometries where the error $|\,\|f(x) - f(y)\| - \|x - y\|\,|$ depends on $\|x - y\|$ and $\|f(x) - f(y)\|$. In this paper we will consider a concept of approximate isometries which was introduced by Šemrl [13] in which the error depends only on $\|x - y\|$. Given a function $\varphi\colon [0,\infty) \to [0,\infty)$, a mapping f from a Banach space X into a Banach space Y is called a φ-isometry if the inequality

$$|\,\|f(x) - f(y)\| - \|x - y\|\,| \le \varphi(\|x - y\|)$$

holds for all $x, y \in X$. As we shall see later the following proposition is a direct consequence of the previously mentioned result of Lindenstrauss and Szankowski.

PROPOSITION 1. *Let X, Y be real Banach spaces, let $f\colon X \to Y$ be a surjective φ-isometry with $f(0) = 0$, and let $\varphi_s\colon [0,\infty) \to [0,\infty)$ be defined by $\varphi_s(t) = \sup_{u \le t}\varphi(u)$. If*

$$\int_1^\infty \frac{\varphi_s(t)}{t^2}\, dt < \infty$$

then there exists an isometry U from X onto Y, defined by $U(x) = \lim_{n\to\infty}(f(2^n x)/2^n)$, *such that*

$$\|f(x) - U(x)\| = o(\|x\|) \qquad \text{as } \|x\| \to \infty.$$

The required condition for φ assures the existence of the isometry U, which is relatively close to f if $\|x\|$ is large, but if $\|x\|$ is small f and U can be relatively wide apart (for example, take the function $f\colon \mathbb{R} \to \mathbb{R}$, defined by $f(x) = 2x - 1$ for $0 < x \le 1$ and by $f(x) = x$ otherwise). If we want to obtain an isometry close to a φ-isometry for small $\|x\|$ we must restrict ourselves to more specific functions φ.

In particular, if $\varepsilon, p \ge 0$ then an (ε, p)-isometry $f\colon X \to Y$ is defined by the inequality

$$|\,\|f(x) - f(y)\| - \|x - y\|\,| \le \varepsilon\|x - y\|^p \qquad \text{for all } x, y \in X.$$

Following [13] we define a pair (X, Y) to be p-stable with respect to isometries if there exists a function $\delta\colon [0,\infty) \to [0,\infty)$ with $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ such that for every surjective (ε, p)-isometry $f\colon X \to Y$ there is a surjective isometry $U\colon X \to Y$ satisfying the inequality $\|f(x) - U(x)\| \le \delta(\varepsilon)\|x\|^p$ for all $x \in X$.

Note that it follows from [11] that every pair (X, Y) is 0-stable with respect to isometries if we take $\delta(\varepsilon) = 2\varepsilon$.

Using some more results from the paper of Lindenstrauss and Szankowski [9] we will prove p-stability also in the range $0 < p < 1$ and obtain the following theorem.

THEOREM 1. *Let X and Y be real Banach spaces and let $0 \le p < 1$. There exists a constant $N(p)$, independent of X and Y, such that for every surjective (ε, p)-isometry $f\colon X \to Y$ with $f(0) = 0$ there exists a surjective isometry $U\colon X \to Y$ satisfying*

$$\|f(x) - U(x)\| \le \varepsilon N(p)\|x\|^p \qquad \textit{for all } x \in X.$$

When the target space Y is a real Hilbert space or when $X = Y = \mathscr{L}_r(0,1)$, $1 < r < \infty$, we obtain a kind of stability result even without the assumption of surjectivity. In addition, because every isometry from a real Banach space into a strictly convex real Banach space is affine (compare, for example, Bhatia and Šemrl [1]), the isometry which approximates the given (ε, p)-isometry is linear.

PROPOSITION 2. *Let X be a real Banach space and let Y be a real Hilbert space. Suppose that $\varepsilon \ge 0$ and $0 < p < 1$. Then there exists a constant $K(\varepsilon, p)$, independent of X and Y, such that $\lim_{\varepsilon\to 0}K(\varepsilon, p) = 0$ and for* *every* (ε, p)*-isometry* $f\colon X \to Y$, $f(0) = 0$, *there exists a linear isometry* U: $X \to Y$ *satisfying*

$$\|f(x) - U(x)\| \leq K(\varepsilon, p)\max\{\|x\|^p, \|x\|^{(1+p)/2}\}. \tag{2}$$

Having in mind the definition of an (ε, p)-isometry, it would be more desirable if the right-hand side of inequality (2) depended only on $\|x\|^p$ and not on $\max\{\|x\|^p, \|x\|^{(1+p)/2}\}$. The following example shows that such an improvement is not possible.

EXAMPLE 1. Consider the mapping $f\colon \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(x) = \begin{cases} (x, 0) & \text{if } x \leq 0, \\ \left(x, \varepsilon \max\{x^p, x^{(1+p)/2}\}\right) & \text{if } x > 0, \end{cases} \tag{3}$$

where $0 < p < 1$ and $0 < \varepsilon \leq 2$. This is an (ε, p)-isometry (compare Lemma 2 in the second part of the paper). Assume that there exists a constant M and an isometry $U\colon \mathbb{R} \to \mathbb{R}^2$, such that

$$\|f(x) - U(x)\| \leq M|x|^p. \tag{4}$$

Since each isometry from $\mathbb{R}$ into $\mathbb{R}^2$ is affine [1], inequality (4) for $x \leq 0$ shows that $U(x) = (x, 0)$. From here it follows that $\|f(x) - U(x)\| = \varepsilon x^{(1+p)/2}$ for $x \geq 1$, which contradicts (4).

As an application of Clarkson's inequality [3], we obtain a similar stability result for $\mathscr{L}_r(0,1)$, $1 < r < \infty$.

PROPOSITION 3. *If* $X = Y = \mathscr{L}_r(0,1)$ *where* $1 < r < \infty$, $\varepsilon \geq 0$, *and* $0 < p < 1$ *then there exists a constant* $K(\varepsilon, p, r)$ *satisfying* $\lim_{\varepsilon \to 0} K(\varepsilon, p, r) = 0$, *such that for every* (ε, p)*-isometry* $f\colon \mathscr{L}_r(0,1) \to \mathscr{L}_r(0,1)$, $f(0) = 0$, *there exists a linear isometry* U *which satisfies the inequality*

$$\|f(x) - U(x)\| \leq K(\varepsilon, p, r)\max\{\|x\|^p, \|x\|^{1-(1-p)/s}\},$$

where $s = \max\{r, \frac{r}{r-1}\}$.

If $p = 1$ even very nice spaces are not stable, as the following proposition shows.

PROPOSITION 4. *The pair* $(\mathbb{R}^2, \mathbb{R}^2)$ *is not* 1*-stable*.

It is somewhat surprising that if $p > 1$ then for finite-dimensional Banach spaces a superstability phenomenon occurs.

THEOREM 2. *If* $\varepsilon \geq 0$ *and* $p > 1$ *then every surjective* (ε, p)*-isometry from a finite-dimensional real Banach space* X *onto a finite-dimensional real Banach space* Y *is an isometry*.

Finally, we give an example which shows that this is not necessarily true for nonsurjective mappings.

EXAMPLE 2. Let $\varepsilon > 0$. We will prove (compare Lemma 3) that the mapping $f\colon \mathbb{R} \to \mathbb{R}^2$, defined by

$$f(x) = \frac{1}{\varepsilon}(\cos(\varepsilon x) - 1, \sin(\varepsilon x)), \tag{5}$$

is an $(\varepsilon, 2)$-isometry, but it is clearly not an isometry since it is not even injective.

2. PROOFS

Proof of Proposition 1. Suppose that

$$\int_1^\infty \frac{\varphi_s(t)}{t^2}\, dt < \infty. \tag{6}$$

Then there exists a constant $M(\varphi) > 0$, such that $t < 2(t - \varphi_s(t))$ for every $t > M(\varphi)$. Indeed, if for every positive integer n we could find a $t_n > n$ such that $\varphi_s(t_n) \geq t_n/2$, then we would have

$$\int_{t_n}^{2t_n} \frac{\varphi_s(t)}{t^2}\, dt \geq \int_{t_n}^{2t_n} \frac{\varphi_s(t_n)}{t^2}\, dt = \varphi_s(t_n)\frac{1}{2t_n} \geq \frac{1}{4},$$

which contradicts (6).

Let $\|f(x) - f(y)\| \leq t$. If $\|x - y\| > M(\varphi)$ then

$$\|x - y\| < 2\big(\|x - y\| - \varphi_s(\|x - y\|)\big) \leq 2\|f(x) - f(y)\| \leq 2t,$$

so

$$|\,\|f(x) - f(y)\| - \|x - y\|\,| \leq \varphi_s(2t).$$

If $\|x - y\| \leq M(\varphi)$ then

$$|\,\|f(x) - f(y)\| - \|x - y\|\,| \leq \varphi_s(M(\varphi)).$$

Now let $\|x - y\| \leq t$. Then

$$|\,\|f(x) - f(y)\| - \|x - y\|\,| \leq \varphi(\|x - y\|) \leq \varphi_s(t) \leq \varphi_s(2t).$$

So, if φ_f is given by (1), we have

$$\varphi_f(t) \leq \max\{\varphi_s(M(\varphi)), \varphi_s(2t)\} \qquad \text{for } t \geq 0. \tag{7}$$

Then

$$\int_{M(\varphi)}^{\infty} \frac{\varphi_f(t)}{t^2}\,dt \le \int_{M(\varphi)}^{\infty} \frac{\varphi_s(2t)}{t^2}\,dt = 2\int_{2M(\varphi)}^{\infty} \frac{\varphi_s(t)}{t^2}\,dt < \infty$$

and we can apply the result of Lindenstrauss and Szankowski.

We will use the following lemma several times.

LEMMA 1. *Let X and Y be real Banach spaces. Suppose $\varepsilon \ge 0$, $0 < p \le r < 1$, and $\delta \ge 0$. If $f\colon X \to Y$, $f(0) = 0$, is an (ε, p)-isometry satisfying*

$$\left\| f(x) - \frac{f(2x)}{2} \right\| \le \delta \max\{\|x\|^p, \|x\|^r\} \qquad \text{for all } x \in X$$

then there exists an isometry $U\colon X \to Y$, defined by $U(x) = \lim_{n\to\infty}(f(2^n x)/2^n)$, such that

$$\|f(x) - U(x)\| \le \delta \frac{2^{1-r}}{2^{1-r}-1} \max\{\|x\|^p, \|x\|^r\} \qquad \text{for all } x \in X.$$

Proof. Let us prove inductively that

$$\left\| f(x) - \frac{f(2^n x)}{2^n} \right\| \le \delta \max\{\|x\|^p, \|x\|^r\} \sum_{i=0}^{n-1} \frac{1}{(2^{1-r})^i} \tag{8}$$

for every positive integer n.

By the assumption, inequality (8) holds for $n = 1$. Suppose that it holds for every $i \le n$. Then

$$\begin{aligned}
&\left\| f(x) - \frac{f(2^{n+1}x)}{2^{n+1}} \right\| \\
&\quad \le \left\| f(x) - \frac{f(2^n x)}{2^n} \right\| + \frac{1}{2^n} \left\| f(2^n x) - \frac{f(2 \cdot 2^n x)}{2} \right\| \\
&\quad \le \delta \max\{\|x\|^p, \|x\|^r\} \sum_{i=0}^{n-1} \frac{1}{(2^{1-r})^i} + \frac{1}{2^n} \cdot \delta \max\{\|2^n x\|^p, \|2^n x\|^r\} \\
&\quad \le \delta \max\{\|x\|^p, \|x\|^r\} \sum_{i=0}^{n} \frac{1}{(2^{1-r})^i}.
\end{aligned}$$

Therefore, (8) is true for every positive integer n and it follows that

$$\left\| f(x) - \frac{f(2^n x)}{2^n} \right\| \leq \delta \frac{1}{1 - 1/2^{1-r}} \max\{\|x\|^p, \|x\|^r\}. \tag{9}$$

Now let us show that $\{f(2^n x)/2^n\}_{n=1}^{\infty}$ is a Cauchy sequence for all $x \in X$. Let m and q be positive integers. Then

$$\begin{aligned}\left\| \frac{f(2^{m+q}x)}{2^{m+q}} - \frac{f(2^m x)}{2^m} \right\| &= \frac{1}{2^m} \left\| \frac{f(2^q \cdot 2^m x)}{2^q} - f(2^m x) \right\| \\ &\leq \delta \frac{2^{1-r}}{2^{1-r} - 1} \cdot \frac{1}{2^m} \max\{\|2^m x\|^p, \|2^m x\|^r\} \\ &\leq \delta \frac{2^{1-r}}{2^{1-r} - 1} \cdot \frac{1}{2^{m(1-r)}} \max\{\|x\|^p, \|x\|^r\}.\end{aligned}$$

Since $r < 1$, we can pick a positive integer m, such that the right-hand side of the inequality is arbitrarily small. The mapping U: $X \to Y$ given by

$$U(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$$

is therefore well defined. Let us prove that it is an isometry. Since f is an (ε, p)-isometry,

$$|\,\|f(2^n x) - f(2^n y)\| - 2^n \|x - y\|\,| \leq 2^{np} \varepsilon \|x - y\|^p,$$

and thus

$$\left|\, \left\| \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n} \right\| - \|x - y\| \,\right| \leq \frac{1}{2^{n(1-p)}} \varepsilon \|x - y\|^p$$

for every positive integer n. Since $p < 1$, the limit of the right-hand side of the inequality equals 0. Therefore

$$|\,\|U(x) - U(y)\| - \|x - y\|\,| = 0,$$

and U is an isometry.

If we send n in inequality (9) to ∞, we get

$$\|f(x) - U(x)\| \leq \delta \frac{2^{1-r}}{2^{1-r} - 1} \max\{\|x\|^p, \|x\|^r\}.$$

Proof of Theorem 1. Suppose $\varepsilon > 0$ and $0 < p < 1$. Let $f\colon X \to Y$ with $f(0) = 0$ be a surjective (ε, p)-isometry and let $\varphi_f(t)$ be given by (1). Since $t < 2(t - \varepsilon t^p)$ if $t > (2\varepsilon)^{1/(1-p)}$, we can take $M(\varphi) = (2\varepsilon)^{1/(1-p)}$ in (7) and obtain

$$\varphi_f(t) \le \varepsilon \max\left\{(2\varepsilon)^{p/(1-p)}, 2^p t^p\right\} \qquad \text{for } t \ge 0.$$

Therefore $\varphi_f(t) = o(t)$ as $t \to \infty$ and by [9, Lemma 3] there exists a bijective map $g\colon X \to Y$ such that

$$\|f(x) - g(x)\| \le \frac{(2\varepsilon)^{1/(1-p)}}{2} \qquad \text{for every } x \in X$$

and

$$\begin{aligned}\varphi_g(t) &\le \varepsilon \max\left\{(2\varepsilon)^{p/(1-p)}, 2^p\left(t + (2\varepsilon)^{1/(1-p)}\right)^p\right\} + (2\varepsilon)^{1/(1-p)} \\ &\le \max\left\{(4\varepsilon)^{1/(1-p)}, \varepsilon\left(2^p\left(t + (2\varepsilon)^{1/(1-p)}\right)^p + 2(2\varepsilon)^{p/(1-p)}\right)\right\}\end{aligned}$$

for $t \ge 0$. It follows easily from the proof of [9, Lemma 3] that we may assume $g(0) = 0$. Since $2^{2p} < 2^{1/(1-p)}$ we have, for $0 \le t \le (2\varepsilon)^{1/(1-p)}$,

$$\begin{aligned}&\varepsilon\left(2^p\left(t + (2\varepsilon)^{1/(1-p)}\right)^p + 2(2\varepsilon)^{p/(1-p)}\right) \\ &\qquad \le \varepsilon\left(2^{2p}(2\varepsilon)^{p/(1-p)} + 2(2\varepsilon)^{p/(1-p)}\right) \\ &\qquad < 2^{p/(1-p)}\varepsilon^{1/(1-p)}\left(2^{2p} + 2^{1/(1-p)}\right) \\ &\qquad < 4^{1/(1-p)}\varepsilon^{1/(1-p)}.\end{aligned}$$

And for $t > (2\varepsilon)^{1/(1-p)}$,

$$\varepsilon\left(2^p\left(t + (2\varepsilon)^{1/(1-p)}\right)^p + 2(2\varepsilon)^{p/(1-p)}\right) < \varepsilon t^p\left(2^{2p} + 2\right) < 4^{1/(1-p)}\varepsilon t^p.$$

So

$$\varphi_g(t) \le 4^{1/(1-p)}\varepsilon \max\{\varepsilon^{p/(1-p)}, t^p\} \qquad \text{for all } t \ge 0.$$

Define $\psi(t) = 4^{1/(1-p)}\varepsilon \max\{\varepsilon^{p/(1-p)}, t^p\}$, $t \ge 0$, and note that $\psi(2t) \le 2\psi(t)$. Take any $x \in X$. We will distinguish two cases. First let $\|x\| \ge (2\varepsilon)^{1/(1-p)}$. Then there exists a positive integer m such that

$$\left(2^m\varepsilon\right)^{1/(1-p)} \le \|x\| < \left(2^{m+1}\varepsilon\right)^{1/(1-p)}.$$

Since

$$\frac{\|x\|}{\psi(\|x\|)} = \frac{\|x\|^{1-p}}{4^{1/(1-p)}\varepsilon} < \frac{2^{m+1}\varepsilon}{4\varepsilon} = 2^{m-1}$$

and

$$\frac{\|x\|^p}{2^{np}} \geq \frac{(2^m\varepsilon)^{p/(1-p)}}{2^{np}} > \varepsilon^{p/(1-p)} \qquad \text{if } n \leq m$$

it follows by [9, proposition] that

$$\begin{aligned}\left\|\frac{g(x)}{2} - g\left(\frac{x}{2}\right)\right\| &\leq 19\psi(\|x\|) + \psi\left(\frac{\|x\|}{2}\right) + \cdots + \psi\left(\frac{\|x\|}{2^{m-1}}\right)\\ &\leq 19 \cdot 4^{1/(1-p)}\varepsilon\left(\|x\|^p + \frac{\|x\|^p}{2^p} + \cdots + \frac{\|x\|^p}{2^{(m-1)p}}\right)\\ &\leq 19 \cdot 4^{1/(1-p)}\varepsilon\|x\|^p\frac{2^p}{2^p - 1}.\end{aligned}$$

So

$$\begin{aligned}&\left\|f\left(\frac{x}{2}\right) - \frac{f(x)}{2}\right\|\\ &\quad\leq \left\|f\left(\frac{x}{2}\right) - g\left(\frac{x}{2}\right)\right\| + \left\|g\left(\frac{x}{2}\right) - \frac{g(x)}{2}\right\| + \frac{1}{2}\|g(x) - f(x)\|\\ &\quad\leq \frac{3(2\varepsilon)^{1/(1-p)}}{4} + 19 \cdot 4^{1/(1-p)}\varepsilon\|x\|^p\frac{2^p}{2^p - 1}\\ &\quad\leq \varepsilon\left(\frac{3}{2} + 38\frac{4^{1/(1-p)}}{2^p - 1}\right)\|x\|^p\end{aligned}$$

in this case. Second, if $\|x\| < (2\varepsilon)^{1/(1-p)}$ then $\|x\| \leq 2\varepsilon\|x\|^p$ and

$$\begin{aligned}\left\|f\left(\frac{x}{2}\right) - \frac{f(x)}{2}\right\| &\leq \left\|f\left(\frac{x}{2}\right)\right\| + \frac{1}{2}\|f(x)\|\\ &\leq \frac{\|x\|}{2} + \varepsilon\left\|\frac{x}{2}\right\|^p + \frac{\|x\|}{2} + \frac{1}{2}\varepsilon\|x\|^p\\ &\leq 2\varepsilon\|x\|^p + \frac{3}{2}\varepsilon\|x\|^p\\ &= \frac{7}{2}\varepsilon\|x\|^p.\end{aligned}$$

We proved that, for every $x \in X$,

$$\left\| f(x) - \frac{f(2x)}{2} \right\| \leq \varepsilon L(p)\|x\|^p.$$

It remains to apply Lemma 1 with $p = r$ to obtain the isometry U. The isometry is surjective since $\varphi_s(t) = \varepsilon t^p$, $t \geq 0$, satisfies the assumptions of Proposition 1.

Proof of Proposition 2. Suppose $\varepsilon > 0$ and $0 < p < 1$. Let us estimate $\|f(x) - f(2x)/2\|$. Since f is an (ε, p)-isometry,

$$\|f(x) - f(2x)\|^2 \leq (\|x\| + \varepsilon\|x\|^p)^2,$$

and thus

$$\|f(x)\|^2 + \|f(2x)\|^2 - 2\langle f(x), f(2x)\rangle \leq (\|x\| + \varepsilon\|x\|^p)^2.$$

It follows that

$$\begin{aligned}
2\left\| f(x) - \frac{f(2x)}{2} \right\|^2
&= 2\|f(x)\|^2 + 2\left\| \frac{f(2x)}{2} \right\|^2 - 4\left\langle f(x), \frac{f(2x)}{2} \right\rangle \\
&= \|f(x)\|^2 + \|f(2x)\|^2 - 2\langle f(x), f(2x)\rangle + \|f(x)\|^2 - 2\left\| \frac{f(2x)}{2} \right\|^2 \\
&\leq (\|x\| + \varepsilon\|x\|^p)^2 + \|f(x)\|^2 - 2\left\| \frac{f(2x)}{2} \right\|^2 \\
&\leq 2(\|x\| + \varepsilon\|x\|^p)^2 - 2\left\| \frac{f(2x)}{2} \right\|^2. \qquad (10)
\end{aligned}$$

We distinguish two cases. First let $\|x\| \geq \frac{1}{2}\varepsilon^{1/(1-p)}$. In this case $\|2x\| - \varepsilon\|2x\|^p \geq 0$ so, since f is an (ε, p)-isometry, $\|f(2x)\|^2 \geq (\|2x\| - \varepsilon\|2x\|^p)^2$ and therefore

$$\begin{aligned}
\left\| f(x) - \frac{f(2x)}{2} \right\|^2 &\leq (\|x\| + \varepsilon\|x\|^p)^2 - \left(\|x\| - \frac{1}{2^{1-p}}\varepsilon\|x\|^p \right)^2 \\
&= \|x\|^2 + 2\varepsilon\|x\|^{1+p} + \varepsilon^2\|x\|^{2p} - \|x\|^2 + 2^p\varepsilon\|x\|^{1+p} \\
&\quad - \frac{1}{(2^{1-p})^2}\varepsilon^2\|x\|^{2p} \\
&\leq 4\varepsilon\|x\|^{1+p} + \varepsilon^2\|x\|^{2p}.
\end{aligned}$$

If $\|x\| < 1$ then $\|x\| \le \|x\|^p$ and therefore

$$\left\| f(x) - \frac{f(2x)}{2} \right\| \le \sqrt{\varepsilon(4+\varepsilon)}\,\|x\|^p.$$

On the other hand, if $\|x\| \ge 1$ then $\|x\| \ge \|x\|^p$ and

$$\left\| f(x) - \frac{f(2x)}{2} \right\| \le \sqrt{\varepsilon(4+\varepsilon)}\,\|x\|^{(1+p)/2}.$$

In the second case, when $\|x\| < \frac{1}{2}\varepsilon^{1/(1-p)}$, that is, $\|x\| \le 1/2^{1-p}\varepsilon\|x\|^p$, it follows from (10) that

$$\left\| f(x) - \frac{f(2x)}{2} \right\| \le \|x\| + \varepsilon\|x\|^p \le \frac{1}{2^{1-p}}\varepsilon\|x\|^p + \varepsilon\|x\|^p \le 2\varepsilon\|x\|^p.$$

So, for any $x \in X$ we have the estimate

$$\left\| f(x) - \frac{f(2x)}{2} \right\| \le 2\sqrt{\varepsilon(4+\varepsilon)}\,\max\left\{\|x\|^p, \|x\|^{(1+p)/2}\right\}.$$

We now apply Lemma 1 and obtain

$$K(\varepsilon, p) = \frac{2^{(3-p)/2}}{2^{(1-p)/2} - 1}\sqrt{\varepsilon(4+\varepsilon)}\,.$$

LEMMA 2. *The mapping* (3) *of Example* 1 *is an* (ε, p)*-isometry.*

Proof. We must prove that for the mapping

$$f(x) = \begin{cases} (x, 0) & \text{if } x \le 0, \\ \left(x, \varepsilon \max\{x^p, x^{(1+p)/2}\}\right) & \text{if } x > 0, \end{cases}$$

where $0 < p < 1$ and $0 < \varepsilon \le 2$, the inequality

$$\big|\, \|f(x) - f(y)\| - \|x - y\| \,\big| \le \varepsilon\|x - y\|^p$$

holds for all $x, y \in \mathbb{R}$. To do so, we will use the simple fact that $x^p - y^p < (x-y)^p$ for $0 < p < 1$ and $x > y > 0$. To see this, write $y = tx$ where $0 < t < 1$, and use the fact that $1 - t^p < (1-t)^p$ for $0 < t, p < 1$. Also, note that for $0 < p < 1$ we have $2p < \frac{3p+1}{2} < p + 1$. Without loss of generality we can assume that $x > y$. We distinguish the following cases:

1. In the case $0 \ge x > y$ there is nothing to prove.
2. If $x > 0 \ge y$ we have to show that

$$\sqrt{\left(\varepsilon \max\{x^p, x^{(1+p)/2}\}\right)^2 + (x-y)^2} - (x-y) \le \varepsilon(x-y)^p$$

or, equivalently,

$$\left(\varepsilon\max\{x^p, x^{(1+p)/2}\}\right)^2 + (x-y)^2 \\ \leq (x-y)^2 + 2\varepsilon(x-y)^{1+p} + \varepsilon^2(x-y)^{2p}.$$

This is true since $(\varepsilon\max\{x^p, x^{(1+p)/2}\})^2 = \varepsilon^2 x^{2p}$ for $0 < x < 1$, and $(\varepsilon\max\{x^p, x^{(1+p)/2}\})^2 = \varepsilon^2 x^{1+p}$ for $x \geq 1$.

3. If $1 \geq x > y > 0$ then, since $(x^p - y^p)^2 < (x-y)^{2p}$, it follows that

$$\varepsilon^2(x^p - y^p)^2 + (x-y)^2 < (x-y)^2 + 2\varepsilon(x-y)^{1+p} + \varepsilon^2(x-y)^{2p},$$

and therefore

$$\sqrt{\varepsilon^2(x^p - y^p)^2 + (x-y)^2} - (x-y) < \varepsilon(x-y)^p.$$

4. If $x > 1 \geq y > 0$ we need to show that

$$\sqrt{\varepsilon^2\left(x^{(1+p)/2} - y^p\right)^2 + (x-y)^2} - (x-y) \leq \varepsilon(x-y)^p$$

or, equivalently,

$$\varepsilon^2\left(x^{(1+p)/2} - y^p\right)^2 + (x-y)^2 \\ \leq (x-y)^2 + 2\varepsilon(x-y)^{1+p} + \varepsilon^2(x-y)^{2p}.$$

This is true because

$$\varepsilon^2\left(x^{(1+p)/2} - y^p\right)^2 \leq \varepsilon^2\left(x^{(1+p)/2} - y^{(1+p)/2}\right)^2 < 2\varepsilon(x-y)^{1+p}.$$

5. Finally, if $x > y > 1$ then, since $(x^{(1+p)/2} - y^{(1+p)/2})^2 < (x-y)^{1+p}$, the following holds

$$\varepsilon^2\left(x^{(1+p)/2} - y^{(1+p)/2}\right)^2 + (x-y)^2 \\ < (x-y)^2 + 2\varepsilon(x-y)^{1+p} + \varepsilon^2(x-y)^{2p},$$

and therefore

$$\sqrt{\varepsilon^2\left(x^{(1+p)/2} - y^{(1+p)/2}\right)^2 + (x-y)^2} - (x-y) < \varepsilon(x-y)^p.$$

And so we proved that f is an (ε, p)-isometry.

Proof of Proposition 3. By Lemma 1 it suffices to show that

$$\left\| f(x) - \frac{f(2x)}{2} \right\| \le \delta(\varepsilon)\max\left\{\|x\|^{p}, \|x\|^{1-(1-p)/s}\right\},$$

where $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ and $0 < p \le 1 - \frac{1-p}{s} < 1$. To do this, we will use the following inequality, due to Clarkson [3],

$$\|u+v\|^{s} + \|u-v\|^{s} \le 2\left(\|u\|^{q} + \|v\|^{q}\right)^{s-1},$$

where $u, v \in \mathscr{L}_r(0,1)$, $1 < r < \infty$, and $s = \max\{r, \frac{r}{r-1}\}$, $q = \min\{r, \frac{r}{r-1}\}$. Note that $s \ge 2$ since $r > 1$ and therefore $1 - \frac{1-p}{s} > p$. We follow the proof of Theorem 1 from [2]. Let us denote $2u = f(x)$ and $2v = f(x) - f(2x)$. Then

$$\begin{aligned}
&\left\| f(x) - \frac{1}{2}f(2x) \right\|^{s} \\
&\quad \le 2^{1-q(s-1)}\left(\|f(x)\|^{q} + \|f(x) - f(2x)\|^{q}\right)^{s-1} - \left\| \frac{1}{2}f(2x) \right\|^{s} \\
&\quad \le 2^{1-q(s-1)}\left(2\left(\|x\| + \varepsilon\|x\|^{p}\right)^{q}\right)^{s-1} - \left\| \frac{1}{2}f(2x) \right\|^{s} \\
&\quad = \left(\|x\| + \varepsilon\|x\|^{p}\right)^{s} - \left\| \frac{1}{2}f(2x) \right\|^{s},
\end{aligned} \tag{11}$$

since $q - qs + s = 0$. Also note that, for $0 \le t \le 1$ and $r > 1$,

$$\begin{aligned}
(1+t)^{r} &\le 1 + (2^{r} - 1)t, \\
(1-t)^{r} &\ge 1 - rt.
\end{aligned}$$

We consider two cases. If $\|x\| \ge \varepsilon^{1/(1-p)}$ then

$$\begin{aligned}
\left\| f(x) - \frac{1}{2}f(2x) \right\|^{s} &\le \left(\|x\| + \varepsilon\|x\|^{p}\right)^{s} - \left(\|x\| - \varepsilon 2^{p-1}\|x\|^{p}\right)^{s} \\
&= \|x\|^{s}\left(1 + \varepsilon\|x\|^{p-1}\right)^{s} - \|x\|^{s}\left(1 - \varepsilon 2^{p-1}\|x\|^{p-1}\right)^{s} \\
&\le \varepsilon(2^{s} - 1)\|x\|^{s+p-1} + s\varepsilon 2^{p-1}\|x\|^{s+p-1} \\
&\le 2^{s+1}\varepsilon\|x\|^{s+p-1},
\end{aligned}$$

so

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \le 4\varepsilon^{1/s} \|x\|^{1-(1-p)/s}.$$

If $\|x\| < \varepsilon^{1/(1-p)}$ then from (11) we get

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \le \|x\| + \varepsilon \|x\|^p \le 2\varepsilon \|x\|^p.$$

Proof of Proposition 4. To prove the proposition, we will modify an example from the paper of Lindenstrauss and Szankowski [9]. Let f_t: $\mathbb{R}^2 \to \mathbb{R}^2$ be the rotation around the origin by the angle t. This mapping satisfies the inequality

$$\|(f_{t+s} - f_t)(x)\| = 2\|x\| \left| \sin\left(\frac{s}{2}\right) \right| \le s\|x\|$$

for all $s \ge 0$ and all $x \in \mathbb{R}^2$. We need another auxiliary function τ: $[0, \infty) \to [0, \infty)$, defined by

$$\tau(t) = \begin{cases} 0 & \text{if } 0 \le t < 1, \\ \varepsilon \log t & \text{if } 1 \le t, \end{cases}$$

where $\varepsilon > 0$. Let us show that, for every $t, s \ge 0$,

$$t(\tau(t+s) - \tau(t)) \le \varepsilon s.$$

We distinguish two cases. If $t \ge 1$ then

$$\varepsilon \log(t+s) - \varepsilon \log t = \varepsilon \log\left(1 + \frac{s}{t}\right) \le \varepsilon \frac{s}{t}.$$

If $0 \le t < 1$ then

$$\tau(t+s) - \tau(t) = 0 \qquad \text{for } t + s \le 1$$

and

$$t\varepsilon \log(t+s) < \varepsilon \log(1+s) < \varepsilon s \qquad \text{for } t + s > 1.$$

Let f: $\mathbb{R}^2 \to \mathbb{R}^2$ be the rotation around the origin by the angle $\tau(\|x\|)$:

$$f(x) = f_{\tau(\|x\|)}(x). \tag{12}$$

Let us verify that f is an $(\varepsilon, 1)$-isometry. Without loss of generality we can assume that $\|x\| \ge \|y\|$. Let us denote $t = \|y\|$, $s = \|x\| - \|y\|$ and make the following estimate

$$\|f_{\tau(t+s)}(y) - f_{\tau(t)}(y)\| \le (\tau(t+s) - \tau(t))\|y\| = (\tau(t+s) - \tau(t))t$$
$$\le \varepsilon s \le \varepsilon\|x - y\|.$$

From here it follows that, on the one hand,

$$\begin{aligned}\|f(x) - f(y)\| &= \|f_{\tau(t+s)}(x) - f_{\tau(t)}(y)\| \\ &\le \|f_{\tau(t+s)}(x) - f_{\tau(t+s)}(y)\| + \|f_{\tau(t+s)}(y) - f_{\tau(t)}(y)\| \\ &\le \|x - y\| + \varepsilon\|x - y\|,\end{aligned}$$

while, on the other hand,

$$\begin{aligned}\|f(x) - f(y)\| &= \|f_{\tau(t+s)}(x) - f_{\tau(t)}(y)\| \\ &\ge \|f_{\tau(t+s)}(x) - f_{\tau(t+s)}(y)\| - \|f_{\tau(t+s)}(y) - f_{\tau(t)}(y)\| \\ &\ge \|x - y\| - \varepsilon\|x - y\|,\end{aligned}$$

and therefore

$$|\,\|f(x) - f(y)\| - \|x - y\|\,| \le \varepsilon\|x - y\|.$$

The mapping f is clearly bijective, since it is bijective on every circle with center at the origin.

Let us show that there is no surjective isometry near f. By the Mazur–Ulam theorem [10] every surjective isometry from $\mathbb{R}^2$ onto $\mathbb{R}^2$ is affine. But, since $f(0) = 0$, the isometry must be linear. There are only two kinds of linear isometries from $\mathbb{R}^2$ onto $\mathbb{R}^2$, rotations U_φ around the origin by an angle φ and reflections U_e across a line through the origin in the direction of a nonzero vector e. In the first case we can find for a given angle φ such an $r \in \mathbb{R}$ that f rotates the circle with radius r by the angle $\varphi + \pi$ so that $\|f(x) - U_\varphi(x)\| = 2r$ for $\|x\| = r$. And in the second case we can find for a given line through the origin an $r \in \mathbb{R}$ such that $f(re_p) = re_p$, where e_p is the unit normal vector of the line, so that $\|f(re_p) - U_e(re_p)\| = 2r$. Thus there is no function $\delta\colon [0,\infty) \to [0,\infty)$ with $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ such that

$$\|f(x) - U(x)\| \le \delta(\varepsilon)\|x\|, \qquad x \in \mathbb{R}^2,$$

for some surjective isometry U. The proof of Proposition 4 is completed.

Proof of Theorem 2. Let $\varepsilon > 0$ and $p > 1$. First, we prove that every (ε, p)-isometry f is a nonexpansive mapping. Let $x, y \in X$ and let n be a positive integer. Then

$$\begin{aligned}
&\|f(x)-f(y)\| \\
&\quad=\left\|\sum_{i=1}^{n}\left(f\left(\frac{(n+1-i)x+(i-1)y}{n}\right)-f\left(\frac{(n-i)x+iy}{n}\right)\right)\right\| \\
&\quad\leq \sum_{i=1}^{n}\left\|\frac{(n+1-i)x+(i-1)y}{n}-\frac{(n-i)x+iy}{n}\right\| \\
&\qquad+\varepsilon\sum_{i=1}^{n}\left\|\frac{(n+1-i)x+(i-1)y}{n}-\frac{(n-i)x+iy}{n}\right\|^{p} \\
&\quad=\|x-y\|+n^{1-p}\varepsilon\|x-y\|^{p}.
\end{aligned}$$

Because $p>1$ and n is arbitrary, it follows that

$$\|f(x)-f(y)\|\leq\|x-y\|.$$

Further, let C be a positive constant such that $C<(2\varepsilon)^{-1/(p-1)}$ and let $0<\|x-y\|\leq C$. Then

$$\frac{\|x-y\|-\varepsilon\|x-y\|^{p}}{\|x-y\|}\geq 1-\varepsilon C^{p-1}>\frac{1}{2}$$

and thus

$$\|x-y\|\leq 2\left(\|x-y\|-\varepsilon\|x-y\|^{p}\right)\leq 2\|f(x)-f(y)\|.$$

This implies that f, restricted to any closed ball of radius less than or equal to $\frac{C}{2}$, is injective.

Now let $f\colon X\to Y$ be a surjective (ε,p)-isometry and let $\dim X=n$ and $\dim Y=m$. Note that $n\leq m$. If not, the map $(f,0)\colon X\to Y\times\mathbb{R}^{n-m}$, restricted to an open ball $B(0,\frac{C}{2})$, would be injective and continuous but not open since the image $f(B(0,\frac{C}{2}))\times\{0\}$ is not an open set in $Y\times\mathbb{R}^{n-m}$. This contradicts the domain invariance theorem.

Our next goal is to prove that $n=m$. Because X is finite dimensional, the closed ball $\overline{B}(0,i)$, where i is a positive integer, is compact. And, since f is continuous, the sequence $\{f(\overline{B}(0,i))\}$, where i runs over all positive integers, is a sequence of closed sets. The mapping f is surjective, and thus $\bigcup_i f(\overline{B}(0,i))=Y$ and so, by Baire's theorem, there exists a positive integer k such that $f(\overline{B}(0,k))$ contains an open ball in Y. Because $\overline{B}(0,k)$ is compact the covering $\{B(x,\frac{C}{2})\colon x\in\overline{B}(0,k)\}$ has a finite subcovering $\{B(x_i,\frac{C}{2})\colon i=1,\dots,l\}$ and $\bigcup_{i=1}^{l} f(\overline{B}(x_i,\frac{C}{2}))$ contains an open ball. Again by Baire's theorem, one of the sets $f(\overline{B}(x_i,\frac{C}{2}))$, $i=1,\dots,l$, let us say

$f(\overline{B}(x_1, \frac{C}{2}))$, must contain an open ball B_1. The inverse $(f|_{\overline{B}(x_1, C/2)})^{-1}$, restricted to B_1, is injective and continuous and by the same argument as before, $n \geq m$. We have shown that, if f: $X \to Y$ is a surjective (ε, p)-isometry with $p > 1$ where X and Y are finite-dimensional Banach spaces, then $\dim X = \dim Y$. It follows by the domain invariance theorem that f is open, and therefore it is a local homeomorphism.

Since

$$1 - \varepsilon \|x - y\|^{p-1} \leq \frac{\|f(x) - f(y)\|}{\|x - y\|} \leq 1, \qquad x \neq y,$$

we have

$$\lim_{x \to y} \frac{\|f(x) - f(y)\|}{\|x - y\|} = 1.$$

To see that f is an isometry, it remains to use a theorem of John [8, Theorem IV], stating that every mapping f from an open connected subset of a Banach space into a Banach space which is a local homeomorphism, and for which

$$\limsup_{x \to y} \frac{\|f(x) - f(y)\|}{\|x - y\|} = \liminf_{x \to y} \frac{\|f(x) - f(y)\|}{\|x - y\|} = 1,$$

coincides with an affine isometry of the whole space.

LEMMA 3. *The mapping* (5) *of Example* 2 *is an* $(\varepsilon, 2)$-*isometry, where* $\varepsilon > 0$.

Proof. Without loss of generality we can assume that $x \geq y$. If $x - y < \frac{1}{\varepsilon}$ then

$$\begin{aligned}
&\big| \|f(x) - f(y)\| - \|x - y\| \big| \\
&\quad = \left| \frac{1}{\varepsilon} \sqrt{(\cos(\varepsilon x) - \cos(\varepsilon y))^2 + (\sin(\varepsilon x) - \sin(\varepsilon y))^2} - (x - y) \right| \\
&\quad = \left| \frac{1}{\varepsilon} \sqrt{2 - 2\cos(\varepsilon(x - y))} - (x - y) \right| \\
&\quad = x - y - \frac{2}{\varepsilon} \sin \frac{\varepsilon(x - y)}{2} \\
&\quad = x - y - (x - y) + \frac{\varepsilon^2 (x - y)^3}{2^2 \cdot 3!} - \frac{\varepsilon^4 (x - y)^5}{2^4 \cdot 5!} + \cdots \\
&\quad \leq \varepsilon (x - y)^2.
\end{aligned}$$

Because $\|f(x)-f(y)\| \le \frac{2}{\varepsilon}$ for all $x, y \in \mathbb{R}$, we see that, for $x-y \ge \frac{1}{\varepsilon}$,

$$\big|\|f(x)-f(y)\|-\|x-y\|\big| \le x-y \le \varepsilon(x-y)^2.$$

Remark. Let us conclude the paper by noting that according to Jarosz [7] a surjective $(\varepsilon, 1)$-isometry $f\colon X \to Y$ with $f(0)=0$ satisfies the following estimate

$$\|f(x+y)-f(x)-f(y)\| \le \delta(\varepsilon)(\|x\|+\|y\|) \qquad \text{for } x, y \in X,$$

where $\lim_{\varepsilon \to 0} \delta(\varepsilon)=0$. So, if f is an $(\varepsilon, 1)$-isometry then it is approximately additive in the sense of the above inequality. However, f need not be approximately linear in the sense of Šemrl [12]; that is, a function δ: $[0,\infty) \to [0,\infty)$ with $\lim_{\varepsilon\to 0}\delta(\varepsilon)=0$, such that the inequality

$$\|f(\lambda x+\mu y)-\lambda f(x)-\mu f(y)\| \le \delta(\varepsilon)(|\lambda|\,\|x\|+|\mu|\,\|y\|),$$

where $\lambda, \mu \in \mathbb{R}$, $x, y \in X$, holds, does not necessarily exist. The $(\varepsilon, 1)$-isometry (12) used in the proof of Proposition 4 is an example of a mapping for which there is no such function δ. To see that, choose any vector $e \in \mathbb{R}^2$ with $\|e\|=1$, so that $f(e)=e$. Then there exists an $r \in \mathbb{R}$ such that f rotates re for the angle π and thus $\|f(re)-rf(e)\| = \|-re-re\| = 2r$.

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